HTL Resummation of the Thermodynamic Potential

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Abstract

Starting from the Φ-derivable approximation scheme at leading-loop order, the thermodynamic potential in a hot scalar theory, as well as in QED and QCD, is expressed in terms of hard thermal loop propagators. This nonperturbative approach is consistent with the leading-order perturbative results, ultraviolet finite, and, for gauge theories, explicitly gauge-invariant. For hot QCD it is argued that the resummed approximation is applicable in the large-coupling regime, down to almost twice the transition temperature.

1   Introduction

One of the central issues of the ongoing heavy-ion program is the investigation of highly excited strongly interacting matter. As predicted by Quantum Chromodynamics (QCD), at energy densities of the order of $1\,\mathrm{GeV/fm^3}$ hadron matter will undergo a transition to a state of deconfined quarks and gluons. Despite the asymptotic freedom of QCD, this quark-gluon plasma (QGP) is characterized by a large coupling strength in the regimes of physical interest. Therefore, nonperturbative approaches are required to describe this many-particle system reliably.

This expectation has been demonstrated in the calculation of the thermodynamic potential of the hot QGP. The perturbative expansion, which is known through order $\mathcal{O}(g^5)$ in the coupling [1], shows no sign of convergence even orders of magnitude above the transition temperature $T_c \sim 170\,\mathrm{MeV}$. Instead, with increasing order of the approximation, the results fluctuate more strongly as a function of the coupling (or

the temperature) along with a growing residual dependence on the renormalization scale. These features are not specific to QCD, but are also observed, e. g., in a scalar theory [2], and are presumably related to the asymptotic nature of perturbative expansions.

On the other hand, extensive studies in lattice QCD indicate for both the pure gauge plasma [3], and for systems containing dynamical fermions [4], that the thermodynamic potential Ω, scaled by the interaction-free limit, is a smoothly increasing function of the temperature, and approaches values $\gtrsim 80\%$ of the ideal gas limit at $T \gtrsim 2T_c$. This saturation-like behavior, as well as the rapid change of Ω between T_c and $2T_c$, has been interpreted within a quasiparticle picture [5], assuming that for large coupling the relevant excitations of the plasma can be described by quasiparticles with effective masses depending on the coupling, as known in the perturbative regime. This phenomenological description amounts to a partial resummation of relevant contributions beyond the leading perturbative corrections. Therefore, the quantitative agreement of these models with the available lattice data is an indication that resummations relying on an appropriate quasiparticle structure may indeed lead to improved approximations for the strong-coupling regime.

For a scalar theory, this conjecture is supported by the so-called screened perturbation theory [6], where the conventional perturbative expansion is rearranged by adding and subtracting a mass term to the lagrangian and expanding in the massive propagator, treating the subtraction as an additional interaction. By relating the mass to the self-energy of the particles, important effects of the interaction are taken into account already at the leading order of the reorganized expansion, which indeed shows an improved behavior for large coupling [6, 7]. The same idea of appropriately rearranging the lagrangian is applied in the hard thermal loop (HTL) perturbation theory [8]. Given the noteworthy properties of the HTL Green's functions which, in particular, satisfy fundamental sum rules and Ward identities in gauge theories, they arise as a preferable basis for reorganized expansions. In the context of thermodynamics, the leading (zeroth) order contributions to the pressure have been calculated within the HTL perturbation theory for the hot QGP [9] as well as for the degenerate plasma [10]. These nonperturbative expressions reproduce correctly the zeroth order of the conventional perturbation theory (i. e., the free limit). In addition, they already include effects of Landau damping and of electrical screening, and hence reproduce at finite temperature the so-called plasmon effect of the

order $\mathcal{O}(g^3)$. However, they do not account correctly for the perturbative $\mathcal{O}(g^2)$ contributions, since also the next-to-leading terms in the HTL perturbation theory contribute to that order. Moreover, as a formal point at issue, an ambiguous regularization scheme dependence related to uncompensated ultraviolet divergences which implicitly depend on the temperature or the chemical potential, will only be improved by the next-to-leading order calculation. This is similar to the situation for screened perturbation theory in the scalar case.

A conceptually different approach is the selfconsistent Φ-derivable approximation scheme [11], which has been applied in [12] for the scalar theory. At leading-loop order, this approximation is equivalent to the large-N limit of the scalar $O(N)$-symmetric model [13], and leads to similar results as the screened perturbation theory, without having to drop the ambiguous temperature dependent divergences in the thermodynamic potential. In [14] it was shown in QED that the entropy derived from the leading-loop Φ-derivable thermodynamic potential can be expressed as a simple functional of the dressed propagators, which themselves have to be determined selfconsistently. This was generalized to QCD in Ref. [15], where a nonperturbative result for the QCD entropy has been obtained by approximately evaluating the entropy functional, at the expense of exact selfconsistency, with the HTL propagators. This approach avoids the nontrivial issues related to gauge invariance and renormalization of the resummed propagators, and leads to a physical and formally well-defined approximation: it is gauge invariant, ultraviolet finite and reproduces the perturbative $\mathcal{O}(g^2)$ result (the next-to-leading order is discussed in [15] as well, see also the remarks below). Moreover, complemented with the two-loop running of the coupling strength, the HTL-resummed entropy matches the lattice results in the saturation-like regime, starting at temperatures $T \gtrsim 2\,T_c$.

From the entropy, given as a function of the temperature, the thermodynamic potential can be obtained, up to an integration constant. However, it is interesting to consider the thermodynamic potential itself in the Φ-derivable approach. As a matter of principle, since in the framework of the HTL approximation (applied for the reasons mentioned above) the thermodynamical selfconsistency holds only approximately, different calculations of the same quantity may lead to different results – which should be compared. More importantly, the approximate thermodynamic potential, expressed in terms of the self-energies which have to be determined as a function of the temperature, contains relevant information not given by the cor-

responding expression for the entropy. As will be shown for the scalar theory and conjectured for gauge theories, from the thermodynamic potential one can infer the range of validity of the approximation in the large-coupling regime.

This paper is organized as follows. In section 2, the concept of Φ-derivable approximations is resumed for the scalar theory. This provides, at leading-loop order, a solvable, yet representative, case of reference. Following [12], selfconsistent and approximately selfconsistent results are derived and discussed with emphasis on the extrapolation to large coupling. For gauge theories, the approach is presented in section 3 for the Abelian case before considering QCD in section 4; parallels to the scalar theory are pointed out. The conclusions are summarized in section 5. Explicit expressions of relevant sum-integrals are relegated to the appendix.

2   Scalar field theory

2.1 Φ-derivable approximations

As an exact relation, the thermodynamic potential Ω can be expressed in terms of the full propagator Δ by the (generalized) Luttinger-Ward representation [16],

$$\Omega = \frac{1}{2} \operatorname{tr}[\ln(-\Delta^{-1}) + \Delta\Pi] - \Phi[\Delta] \,, \tag{1}$$

where the trace is taken over all states of the system. The exact self-energy Π, which is related by Dyson's equation to the free propagator Δ_0 and the full propagator by

$$\Delta^{-1} = \Delta_0^{-1} - \Pi \,, \tag{2}$$

can be represented diagrammatically as the series of the dressed one-particle irreducible graphs $\hat{\Pi}_n$ of order n. In the expression (1), the interaction-free contribution Ω_0 is contained in the trace part, as is obvious from expanding $\ln(-\Delta^{-1}) + \Delta\Pi = \ln(-\Delta_0^{-1}) + \ln(1 - \Delta_0\Pi) + \Delta_0\Pi/(1 - \Delta_0\Pi)$ in powers of Π, while the leading-order perturbative correction is entirely due to the Φ contribution. The functional Φ in (1) is given by the skeleton diagram expansion

$$\hat{\Phi} = \sum_n \frac{1}{4n} \operatorname{tr}[\Delta\hat{\Pi}_n] \,. \tag{3}$$

Thus, taking into consideration the combinatorial factors related to the number of propagators in each graph of this expansion, the self-energy is obtained diagrammatically from $\hat{\Phi}$ by opening one of the propagator lines in the individual graphs,

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i. e.,

$$\Pi = 2\,\frac{\delta\Phi}{\delta\Delta}\,. \tag{4}$$

Consequently, the thermodynamic potential (1) considered as a functional of Δ is stationary at the exact propagator defined by the solution of the implicit functional equation (4),

$$\frac{\delta\Omega}{\delta\Delta} = 0\,. \tag{5}$$

This fundamental relation expresses the thermodynamical selfconsistency between microscopic and macroscopic properties of the system [11].

Commencing from this exact framework, selfconsistent approximations of the thermodynamic potential can be derived [11]: truncating the complete skeleton expansion of the functional Φ at a certain loop order, and calculating selfconsistently the approximate self-energy analogously to (4), yields an approximation of the thermodynamic potential which is still stationary with respect to variations of the resummed propagator. In this Φ-derivable approximation scheme, appropriate sets of diagrams contributing to the self-energy and to the thermodynamic potential are resummed, in such a way that thermodynamical properties can be calculated from the thermodynamic potential, utilizing thermodynamical relations, or directly from the Green's functions, which leads to the same, approximate result.

2.2 Selfconsistent leading-loop resummation

With the interaction of the scalar particles described by $\mathcal{L}_i = (-g_0^2/4!)\,\phi^4$, the functional Φ and the corresponding self-energy are given at the leading-loop order in the skeleton expansion, with explicit symmetry factors, by

$$\Phi_{ll} \;\; = \;\; 3\,\bigcirc\!\bigcirc\,,$$

$$\Pi_{ll} \;\; = \;\; 12\,\bigcirc \;=\; 12\left(\frac{-g_0^2}{4!}\right) I(\Pi_{ll}, T)\,. \tag{6}$$

This truncation of the skeleton expansions is equivalent to considering the leading contributions of the $1/N$ expansion in the scalar $O(N)$-symmetric theory [13], hence providing exact results in the limit $N \to \infty$. In terms of the conventional perturbation theory, it amounts to a complete resummation of the so-called super-daisy diagrams [17].

Since in (6) the 2-point contribution is local, Π_{ll} is just a mass term, which reduces the implicit functional equation to a transcendental, yet nontrivial, relation. In the imaginary time formalism, regularizing the spatial momentum integrals in $d = 3 - 2\epsilon$ dimensions in the $\overline{\text{MS}}$ scheme, the trace over the momentum $K = (k_0, k)$ is defined as

$$\text{tr} = \sum_{k_0} \!\!\!\!\!\int = \int_{k^d} T \sum_{k_0} , \quad \int_{k^d} = \left(\frac{e^\gamma \bar{\mu}^2}{4\pi} \right)^\epsilon \int \frac{d^d k}{(2\pi)^d} ,$$

with the renormalization scale $\bar{\mu}$ and Euler's constant γ, and where $k_0 = i\, 2n\pi T$ are the bosonic Matsubara frequencies. The function

$$I(M^2, T) = \sum \!\!\!\!\!\int \frac{1}{K^2 - M^2} = I^0(M^2) + I^T(M^2, T) \tag{7}$$

is decomposed into two parts: a contribution I^0 which is not explicitly temperature dependent (below, M will depend on T), and associated to the quantum fluctuations of the vacuum, and a part I^T due to the thermal medium, which are both given in the appendix. While the thermal fluctuations are cut off by the Bose distribution function $n_b(x) = [\exp(x) - 1]^{-1}$, the ultraviolet divergence of the vacuum contribution is apparent in a pole term $\propto M^2/\epsilon$. In the gap equation (6), identifying M^2 with the mass term $m_0^2 + \Pi_{ll}$ of the propagator Δ_{ll}, this divergence is temperature dependent. The contribution $\propto m_0^2/\epsilon$ is absorbed in the physical mass by the conventional mass renormalization[1]. Focusing, to simplify the discussion, on the temperature dependent part $\propto \Pi_{ll}/\epsilon$, only the case of massless particles (as relevant for the ultrarelativistic gauge plasmas studied below) is considered in the following, with $\Delta_0^{-1}(k_0, k) = K^2 = k_0^2 - k^2$.

The thermal divergence in (6) requires the renormalization of the bare coupling g_0. The physical coupling can be related to the vacuum scattering amplitude. In the approximation considered, resumming the set of chain diagrams,

the renormalized coupling at the momentum scale $s = P^2$ is determined by

$$g^2(s) = g_0^2 + 12 \left(\frac{-g_0^2}{4!} \right) L(s)\, g^2(s) ,$$

$$L(P^2) = \int_{K^{d+1}} \frac{1}{K^2} \frac{1}{(P-K)^2} = \frac{1}{16\pi^2} \left[\frac{1}{\epsilon} + \ln \frac{\bar{\mu}^2}{-P^2} + 2 \right]. \tag{8}$$

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[1]In the present approach, the mass counter term is obtained from a 'counter loop' contribution $\delta\Phi = \frac{1}{2} \text{tr}\, \Delta (\delta m)^2$ to Φ, which does not affect the thermodynamic potential.

Omitted here are the crossed diagrams in the scattering amplitude; these induce graphs with a different topology in the expansion of the thermodynamic potential and the self-energy[2],
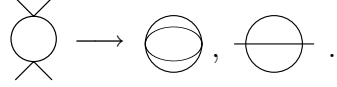
Expressed in terms of the renormalized coupling, the Dyson equation reads

$$\Pi_{ll} = \frac{g^2(s)}{2}\left[-I^T(\Pi_{ll},T) + \frac{\Pi_{ll}}{16\pi^2}\left(\ln\frac{\Pi_{ll}}{-s}+1\right)\right].\tag{9}$$

The right hand side of this implicit gap equation for Π_{ll} is finite and independent of $\bar{\mu}$. Describing the system under consideration by the value of $g^2(s)$ at a specific scale s, and taking into account that $g^{-2}(s') = g^{-2}(s) - \ln(s'/s)/(32\pi^2)$ according to eqn. (8), the solution of (9) is invariant under the rescaling $s \to s'$. In weak coupling, by expanding $I^T(\Pi_{ll},T)$ in small Π_{ll}/T^2, the resummed leading loop self-energy reproduces the perturbative result

$$\Pi_{\mathrm{pert}} = \frac{g_0^2 T^2}{4!}\left[1 - \frac{3}{\pi}\frac{g_0}{\sqrt{4!}} + \dots\right]\tag{10}$$

up to next-to-leading order, where the coupling is still unrenormalized. In the gap equation (9), the selfconsistent resummation leads to a nontrivial interplay between vacuum and thermal fluctuations. In particular, as discussed in detail in [13], for not too large coupling the gap equation has two solutions, see Fig. 1. While the smaller one is associated with the perturbative approximation, the second, tachyonic, solution is exponentially large for small g^2 and thus of no physical relevance. For the choice $-s = T^2$ and couplings larger than the value $g^2_{\mathrm{max}} \sim 100$, the gap equation has no solution. Clearly, the present approximation is physically justifiable only for couplings below g^2_{max}, where both solutions of the gap equation are not of the same order as the maximal value $\Pi_{ll}^{\mathrm{max}} \sim 4.2\,T^2$. The perturbative result (10), on the other hand, shows the typical features of asymptotic expansions: with higher

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[2] I. e., the contribution of the set of super-daisy diagrams to the self-energy is finite after renormalizing the coupling in the same class of graphs. As a consequence of the present approximation, therefore, the running of g^2 is determined by a β function which differs from the perturbative result by a factor of $1/3$. It agrees with the β function of the scalar $O(N)$ model in the large-N limit considered in [13], where the crossed diagrams in the scattering amplitude are suppressed. Hence, g^2 as defined in (8) is related to the $N = 1$ running coupling only at a single value of the scale.
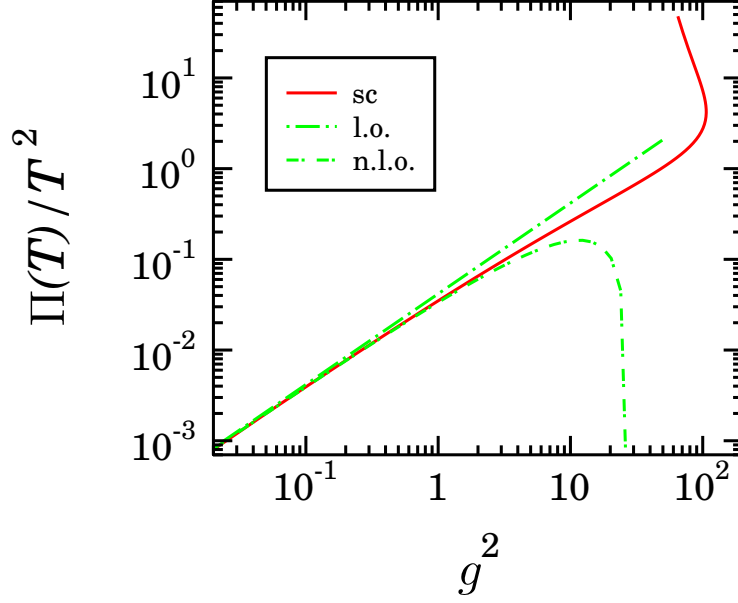
Figure 1: The self-energy, in the scalar theory, as a function of the coupling. Compared are the selfconsistent (sc) solution of the gap equation, with the coupling renormalized at the scale $-s = T^2$, and the leading (l.o.) and next-to-leading order (n.l.o.) perturbative results depending on the bare coupling.

order, the accuracy of the approximation is improved only in a limited range, which becomes smaller with increasing order, of the expansion parameter.

Having solved the gap equation for a given value of $g^2(s)$, the approximation Ω_{ll} of the thermodynamic potential can be calculated as a function of Π_{ll}. From the vacuum part J^0 of the function

$$J(M^2, T) = \frac{1}{2} \sum\!\!\!\!\!\!\!\int \ln(-\Delta_0^{-1} + M^2) = J^0(M^2) + J^T(M^2, T), \qquad (11)$$

which is given explicitly in the appendix, it appears that the contribution

$$(\Omega + \Phi)_{ll} = \frac{1}{2} \sum\!\!\!\!\!\!\!\int [\ln(-\Delta_{ll}^{-1}) + \Delta_{ll}\Pi_{ll}] = \frac{\Pi_{ll}^2}{64\pi^2}\left[-\frac{1}{\epsilon} + \frac{2}{\epsilon}\right] + \text{finite terms} \qquad (12)$$

to the thermodynamic potential contains thermal divergences which are only partly compensated between the $\ln(-\Delta^{-1})$ and the $\Delta\Pi$ term, as indicated by the bracket. In fact, they are cancelled by the remaining Φ contribution. With the two-loop Φ

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functional evaluated with the selfconsistent propagator,

$$\Phi_{ll}[\Delta_{ll}] = \frac{1}{4} \sum\!\!\!\!\!\!\int \Delta_{ll}\Pi_{ll} \,, \tag{13}$$

where the renormalization of the coupling has been taken into account, the re-summed thermodynamic potential takes the form

$$\Omega_{ll} = \frac{1}{2} \sum\!\!\!\!\!\!\int \left[\ln(-\Delta_{ll}^{-1}) + \frac{1}{2}\Delta_{ll}\Pi_{ll} \right] , \tag{14}$$

which is, indeed, finite and independent of the renormalization scale. In terms of the functions (7) and (11), the pressure $p = -\Omega$ (the volume is set to unity) is, in the selfconsistent approximation,

$$p_{ll} = -J^T(\Pi_{ll}, T) - \frac{1}{4}\Pi_{ll}\, I^T(\Pi_{ll}, T) + \frac{\Pi_{ll}^2}{128\pi^2} \,. \tag{15}$$

The first term is the pressure of a free gas of quasiparticles with mass $\Pi_{ll}^{1/2}$, while the other terms represent the interactions among these quasiparticles. The last term is only implicitly temperature dependent. It stems from the vacuum part of the resummed contributions, and leads at larger Π_{ll} to an increasing ratio of p_{ll} to the free pressure $p_0 = \pi^2 T^4/90$, as shown in Fig. 2. This behavior, however, is physically not relevant. Instead, it indicates a breakdown of the approximation: as a result of the stationary condition (5) of the thermodynamic potential, the value of the self-energy at the point where p_{ll} has its minimum coincides with the maximal solution Π_{ll}^{max} of the Dyson equation,

$$\Pi_{ll}^{\mathrm{max}} \sim 4.2\,T^2, \quad p_{ll}^{\mathrm{min}} = p_{ll}(\Pi_{ll}^{\mathrm{max}}) \sim 0.73\,p_0\,. \tag{16}$$

Therefore, the behavior of the thermodynamic potential (14) provides the same strong criterion for the applicability of the approximation as that of the solution of the gap equation, which, in both cases, is due to the interplay between the resummed contributions of the thermal and the vacuum fluctuations.

It remains to note that similarly to the expression (9) for the dressed self-energy, the approximation (15) of the pressure, which resums terms of all orders in the coupling, agrees with the perturbative result

$$p^{pert} = p_0 \left[1 - \frac{15}{8\pi^2}\frac{g_0^2}{4!} + \frac{15}{2\pi^3}\left(\frac{g_0^2}{4!}\right)^{3/2} + \dots \right] \tag{17}$$
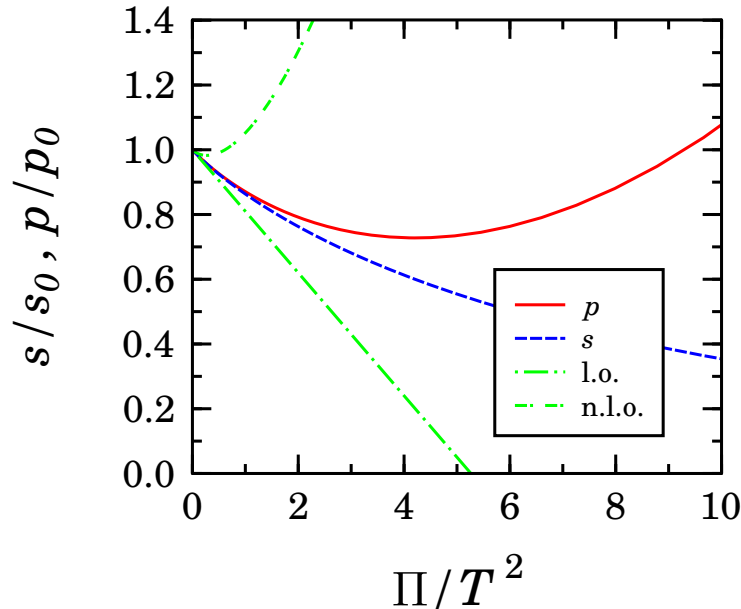
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Figure 2: The selfconsistent pressure and the entropy (in units of the free values) in the scalar theory, as functions of the resummed self-energy. These results coincide with the HTL approximations of p and s as functions of $\Pi^\star$. Physical relevance can be attributed to these approximations only in the regime where p is a decreasing function of Π, see text. Also shown for comparison are the leading and next-to-leading order perturbative results (for both p and s) depending on $g_0^2/4! = \Pi^\star/T^2$.

up to next-to-leading order. As asymptotic expansions, the perturbative results start to fluctuate with increasing order, and the range of predictability is rather small already for the next-to-leading order expression, as apparent in Fig. 2. The resummed approximation, on the other hand, continues smoothly into the large-coupling regime, as expected on physical grounds for the exact result.

Shown also in Fig. 2 is the entropy (density) related to the pressure by the thermodynamic relation $s = -d\Omega/dT = dp/dT$. In the leading-loop approximation, from eqn. (15),

$$s_{ll} = -\left.\frac{\partial J^T}{\partial T}\right|_{\Pi_{ll}} - \frac{\Pi_{ll}}{4}\left.\frac{\partial I^T}{\partial T}\right|_{\Pi_{ll}} + \left.\frac{\partial p_{ll}}{\partial \Pi_{ll}}\right|_T \frac{d\Pi_{ll}}{dT},$$

the second and the third terms cancel for the selfconsistent solution of the gap

equation, hence

$$s_{ll} = -\left.\frac{\partial J^T(\Pi_{ll}, T)}{\partial T}\right|_{\Pi_{ll}}.$$ (18)

Contrary to the pressure (15), the entropy s_{ll}, as a measure of the population of the phase space, is equivalent to that of a system of free quasiparticles with mass $\Pi_{ll}^{1/2}$ [12]. Therefore, s_{ll} is a monotonically decreasing function of the self-energy, without any indication of the breakdown of the approximation. However, according to the preceding considerations, only values of the entropy larger than $s_{ll}^{\min} \sim 0.60\, s_0$ are justified in the present approach.

Alternatively (not specific to the scalar theory), the entropy derived in the self-consistent approach with the two-loop approximation of Φ can also be evaluated from a functional, independent of Φ, of the dressed propagator [14, 15]. This direct calculation benefits from the observation that the entropy is, in contrast to the pressure, sensitive only to the thermal excitations of the system and, thus, a manifestly ultraviolet finite quantity. Formally, this is related to the fact that the function $p_{ll}(\Pi_{ll})$ indicates a breakdown of the approximation, but $s_{ll}(\Pi_{ll})$ does not. This statement does not contradict the thermodynamical relation between the pressure and the entropy since, to reconstruct the pressure p_{ll} from eqn. (18), the selfconsistent self-energy has to be known as a function of the temperature. In that sense, the pressure expressed in terms of the self-energy provides relevant information not contained in the entropy – which makes the pressure an interesting quantity to consider for cases where the self-energy cannot be resummed selfconsistently, as examined in the following.

2.3 Approximately selfconsistent resummations

With regards to gauge theories considered below, it is instructive to study an additional approximation within the leading-loop Φ-derivable approach, by solving the gap equation only approximately.

The leading-order perturbative contribution of the resummed self-energy (6) is obtained from the tadpole graph with the bare propagator and agrees with the HTL approximation [8] (denoted by a star),

$$\Pi^\star = \frac{g_0^2 T^2}{4!}.$$ (19)

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At this level of approximation, no renormalization is required (the perturbative vacuum divergence vanishes in dimensional regularization). As put forward in [15], a resummed approximation of the entropy can be obtained from the aforementioned entropy functional evaluated with the dressed propagator

$$\Delta^\star = [\Delta_0^{-1} - \Pi^\star]^{-1} \,, \tag{20}$$

making use of the observation that this functional does not dependent on Φ_{ll} for any *ansatz* of the propagator. The resulting approximation of the entropy is then given again by the free quasiparticle expression, and the only difference to the self-consistently resummed approximation (18) is the quasiparticle mass, which is now determined by (19). In contrast to the selfconsistent result, however, this approximation reproduces the perturbative series only to order $\mathcal{O}(g_0^2)$. The next-to-leading correction is underestimated by a factor of $1/4$ since, diagrammatically, the set of daisy (ring) diagrams is included correctly only after the resummation of the self-energy. Indeed, replacing $\Pi^\star$ by the next-to-leading order perturbative expression (10), as considered in [15], yields the correct $\mathcal{O}(g_0^3)$ term for the entropy. It is recalled, however, that the $\mathcal{O}(g_0^3)$ contribution to the self-energy has its origin in the screening of the dressed propagator, and that the related thermal divergence $\sim g_0^2(g_0 T)^2$, which cancels in the selfconsistent gap equation, has to be dropped here in the spirit of perturbation theory[3].

Returning now to the thermodynamic potential, one could approximately evaluate the functional (1) with the two-loop contribution to Φ and the HTL propagator (20). This approximation indeed resums the set of the individually infrared divergent daisy diagrams, and thus reproduces the perturbative result up to the order $\mathcal{O}(g_0^3)$ for the plasmon term. However, $\Phi_{ll}[\Delta^\star]$ does not combine with the second term of $\frac{1}{2}\operatorname{tr}[\ln(-\Delta^{\star\,-1}) + \Delta^\star\Pi^\star]$ as for the selfconsistent approximation, so the total expression contains a temperature dependent divergence $\propto (\Pi^\star)^2 \propto (g_0 T)^4$, and is therefore relevant only as a perturbative expansion up to order $\mathcal{O}(g_0^3)$.

Alternatively, one can approximate the Φ contribution by

$$\Phi^\star = \frac{1}{4} \sum\!\!\!\!\!\!\int \Delta^\star\Pi^\star \,, \tag{21}$$

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[3]On the other hand, to obtain improved quantitative results for the entropy at larger coupling strength, the authors of Ref. [15] used a Padé approximation of the next-to-leading order self-energy, which contains higher powers of g_0.

which resembles the selfconsistent contribution in (13). As required for an appropriate approximation (since the leading-order contribution to the thermodynamic potential arises from the Φ functional), $\Phi^\star$ agrees to order $\mathcal{O}(g_0^2)$ with Φ_{ll}. Furthermore, the resulting approximation for the thermodynamic potential,

$$\Omega^\star = \frac{1}{2} \sum\!\!\!\!\!\!\int_f \left[\ln(-\Delta^{\star\,-1}) + \frac{1}{2} \Delta^\star \Pi^\star \right], \tag{22}$$

is obviously ultraviolet finite because it has the *same* functional dependence on $\Pi^\star$ as the selfconsistent approximation $\Omega_{ll}(\Pi_{ll})$. The expansion of the pressure for small $\Pi^\star/T^2 = g_0^2/4!$ is

$$p^\star = p_0 \left([1-0] - [2-1] \frac{15}{8\pi^2} \frac{\Pi^\star}{T^2} + [1-3/4] \frac{15}{2\pi^3} \left(\frac{\Pi^\star}{T^2} \right)^{3/2} + \ldots \right), \tag{23}$$

(the brackets indicate the contributions of the two terms in eqn. (22)). This reproduces the perturbative result (17) to order $\mathcal{O}(g_0^2)$, but underestimates the $\mathcal{O}(g_0^3)$ term by a factor of $1/4$, for the same reason and with the same implications as given above for the entropy. Even so, the approximation (22) is physically significant, since the self-energy (in general, the mass scale of the self-energy) is in principle a measurable quantity. It is more important, however, that it also provides a restricting criterion for the validity of the approach to approximate the thermodynamic potential in terms of the HTL propagator. Clearly, the approximation $\Omega^\star(\Pi^\star)$ cannot be justified in a regime of strong coupling where even the selfconsistent approximation is not physical. In other words, the minimum of $p^\star(\Pi^\star)$ provides, even without having at hand the selfconsistent solution of the gap equation, a strong limit of applicability of the approximately selfconsistent approach. This fact will be important in the following discussions of gauge theories.

3   QED

In this section, an ultrarelativistic electron-positron plasma, described by QED, is considered at temperatures much larger than the electron mass.

For gauge theories, an additional requirement for a formally consistent approximation of a physical quantity is gauge invariance. In the Φ-derivable approach (unless solved exactly), by resumming thermodynamically selfconsistent sets of graphs, this requirement is, in general, not satisfied, since the two-point functions are distinguished in the hierarchy of Green's functions. Besides, it is not obvious how a

renormalization of the coupling constant can be accomplished technically, to account for the thermal divergences in the resummed Dyson equations. However, it will turn out that the leading contributions to the resummed thermodynamic potential arise as in the scalar theory from the HTL parts of the propagators, which are, in fact, gauge invariant, and renormalized by the usual vacuum counter-terms. Therefore, an explicitly gauge-independent nonperturbative approximation of Ω can be derived, which exhibits analogous features as the approximately selfconsistent resummation in the scalar case. Later I will argue that this approximation allows one to conjecture about the large-coupling behavior of the leading-loop resummation.

The exact thermodynamic potential can be expressed as a functional of the photon propagator D and the electron propagator S, which are by Dyson's equation related to the respective self-energies Π and Σ, as[4]

$$\Omega = \frac{1}{2} \operatorname{Tr}[\ln(-D^{-1}) + D\Pi] - \operatorname{Tr}[\ln(-S^{-1}) + S\Sigma] - \Phi[D, S]. \qquad (24)$$

In the boson contribution, the trace is taken over the four-momentum as well as over the Lorentz structure, while in the fermion part the trace includes the spinor indices. The functional $\Phi[D, S]$ is given by the series of skeleton graphs with exact propagators. It is related to the self-energies by

$$\Pi = 2 \frac{\delta \Phi}{\delta D}, \quad \Sigma = -\frac{\delta \Phi}{\delta S}, \qquad (25)$$

so the thermodynamic potential is stationary at the full propagators.

By the projectors $\mathcal{P}^T_{\mu\nu} = g_{\mu\nu} - K_\mu K_\nu/K^2 - \mathcal{P}^L_{\mu\nu}$ and $\mathcal{P}^L_{\mu\nu} = -\tilde{K}_\mu \tilde{K}_\nu/K^2$, where $\tilde{K} = [K(Ku) - uK^2]/[(Ku)^2 - K^2]^{1/2}$ and u is the medium four-velocity, the inverse photon propagator is decomposed into the transverse (T) and the longitudinal (L) part as well as the covariant gauge-fixing term,

$$D^{-1}_{\mu\nu}(K) = \sum_{i=T,L} \mathcal{P}^i_{\mu\nu} \Delta^{-1}_i(K) + \frac{1}{\xi} K_\mu K_\nu, \quad \Delta^{-1}_i = \Delta^{-1}_0 - \Pi_i.$$

Introducing the 'projectors' $\mathcal{P}_\pm = \frac{1}{2}(\not{K} \pm \tilde{\not{K}})$ (the index denotes the ratio of chirality to helicity), the fermion propagator can be written in a similar way as

$$S(K) = \sum_{i=\pm} \mathcal{P}_i \Delta_i(K), \quad \Delta^{-1}_i = \Delta^{-1}_0 - \Sigma_i.$$

---

[4]Depending on the gauge, the contribution of the Abelian ghost fields, which compensate the non-transverse degrees of freedom, is included implicitly.

In terms of the scalar propagators Δ_i, with the degeneracy factors $d_T = d-1$, $d_L = 1$ and $d_\pm = (d+1)/2$, the first two contributions in eqn. (24) read

$$
\Omega + \Phi = \frac{1}{2} \sumint \left\{ \sum_{i=T,L} d_i \left[\ln(-\Delta_i^{-1}) + \Delta_i \Pi_i \right] - \ln(-\Delta_0^{-1}) \right\}
$$
$$
- \sumint \left\{ \sum_{i=\pm} d_i \left[\ln(-\Delta_i^{-1}) + \Delta_i \Delta_0^{-1} \right] - d_\pm \ln(-\Delta_0^{-1}) \right\}, \qquad (26)
$$

where the subtractive contribution of the Abelian ghost fields, which otherwise decouple, is included in the boson term.

To leading-loop order, ignoring for a moment the question of gauge dependence, the Φ functional and the self-energies are determined by

$$
\Phi_{ll} = \frac{1}{2} \; \raisebox{-1em}{\rule{0pt}{0pt}} \quad , \quad \Pi_{ll} = \quad \raisebox{-1em}{\rule{0pt}{0pt}} \quad , \quad \Sigma_{ll} = - \quad \raisebox{-1em}{\rule{0pt}{0pt}} \quad . \qquad (27)
$$

In the following, as motivated above, the self-energies are approximated by their HTL contributions. This approximation, as a matter of fact, undermines the self-consistency of the Φ-derivable approach. As discussed for the scalar theory, without the resummation of the self-energies in (27), the set of ring diagrams contributing to the thermodynamic potential at next-to-leading order is included only incompletely, and the coupling constant e remains bare. Even more important, it is not obvious *a priori* that a physically meaningful approximation of Ω can be formulated in terms of the HTL propagators. These are derived for soft momenta, much smaller than the temperature, whereas thermodynamics is sensitive to the momentum scale T. It will turn out, however, that the dominant contributions originate from the vicinity of the light cone, where the HTL approximation is appropriate, even at large momenta.

In the HTL approximation, the self-energies are given by [8]

$$
\Pi_T^\star = M_\gamma^2 + \tilde{\Pi}, \quad \Pi_L^\star = -2\tilde{\Pi}, \quad \tilde{\Pi}(k_0, k) = M_\gamma^2 \frac{K^2}{k^2} \left[1 + \frac{k_0}{2k} \ln \frac{k_0 - k}{k_0 + k} \right],
$$
$$
\Sigma_\pm^\star = \frac{1}{2} M_e^2 \pm \tilde{\Sigma}, \quad \tilde{\Sigma}(k_0, k) = \frac{M_e^2}{2} \left[\frac{k_0}{k} + \frac{K^2}{2k^2} \ln \frac{k_0 - k}{k_0 + k} \right]. \qquad (28)
$$

The quantities

$$
M_\gamma^2 = \frac{e^2 T^2}{6}, \quad M_e^2 = \frac{e^2 T^2}{4} \qquad (29)
$$

are referred to as the asymptotic masses (squared) of the transverse photon and the electron particle excitations, respectively, since their dispersion relations approach mass shells for momenta $k \gtrsim eT$. The longitudinal photon (plasmon) mode and the electron hole (plasmino) excitation, on the other hand, possess an exponentially vanishing spectral strength for $k \gtrsim eT$ when approaching the light cone.

The contribution (26) to the thermodynamic potential can be approximated by evaluating the functional with the HTL propagators (see appendix and, for details of the calculation, [9, 18]). It is worth pointing out that the $\ln(-\Delta_i^{-1})$ and the $\Delta_i \Delta_0^{-1}$ contributions of the fermions, summed over $i = \pm$, are ultraviolet finite individually, while in the respective boson contributions only the most severe thermal divergences $\propto M_\gamma^4/\epsilon^2$ cancel between the transverse and longitudinal terms. Surprisingly, although now the self-energies are nonlocal and have imaginary parts, the structure of the overall divergence is in an apparent analogy to the scalar case (12),

$$(\Omega + \Phi)^\star = \frac{M_\gamma^4}{32\pi^2} \left[-\frac{1}{\epsilon} + \frac{2}{\epsilon} \right] + \text{finite terms}, \tag{30}$$

where the terms in the bracket originate from the first and second contribution in $\frac{1}{2}\text{Tr}[\ln(-D^{-1}) + D\Pi]$. Again, this divergence is temperature dependent and, thus, expected to be cancelled by the remaining Φ contribution.

In a selfconsistent approach, the Φ_{ll} functional evaluated with the selfconsistent propagators could be expressed as traces over the self-energies,

$$\Phi_{ll} = \frac{1}{2}\text{Tr}D_{ll}\Pi_{ll} = -\frac{1}{2}\text{Tr}S_{ll}\Sigma_{ll}, \tag{31}$$

analogously to the identity (13) in the scalar theory. However, the naive replacement of selfconsistent propagators and self-energies by their HTL approximations, as in (21) for the scalar case, leads to different results for the bosonic and the fermionic traces even at order $\mathcal{O}(e^2)$, since the expressions are dominated by hard momenta where contributions neglected in the HTL approximation become important. Even so, there is an unique combination of the two traces which keeps track of these terms. Denoting the photon momentum in $\frac{1}{2} \bigodot$ by K and the fermion momenta by $Q_{1,2}$, this diagram (with bare propagators for the $\mathcal{O}(e^2)$ contribution) can be represented as a double sum-integral over an expression with a numerator $N = K^2 - Q_1^2 - Q_2^2$. Closing the external legs of the boson self-energy in the HTL approximation amounts to neglecting the term K^2 in N. Tracing over the fermion HTL self-energy, on the other hand, neglects one of the Q^2 terms. Thus, all terms are accounted for twice

in the sum over all three possibilities to approximate one of the momenta as soft. Accordingly, the specific combination

$$\Phi^\star = \frac{1}{4}\operatorname{Tr}D^\star\Pi^\star - \frac{1}{2}\operatorname{Tr}S^\star\Sigma^\star \tag{32}$$

is equivalent, at leading order, to the Φ contribution (31). As shown in the following, this unique approximation indeed leads to a well-defined resummed approximation of the thermodynamic potential.

It is first emphasized that the complete expression resulting from the HTL approximation of eqn. (26) and (32), which can be written in a compact form as

$$\Omega^\star = \frac{1}{2}\operatorname{Tr}\left[\ln(-D^{\star\,-1}) + \frac{1}{2}\,D^\star\Pi^\star\right] - \operatorname{Tr}\left[\ln(-S^{\star\,-1}) + \frac{1}{2}\,S^\star\Sigma^\star\right], \tag{33}$$

is analogous to the corresponding expression (22) in the scalar theory. After the precedent discussion of the structure of the divergences of the individual terms in the contribution (30), this implies that $\Omega^\star$ is an ultraviolet finite quantity and explicitly independent of the regularization scale. In the interaction-free limit, it reduces to the thermodynamic potential $\Omega_0 = -(d_T + \frac{7}{8}\,2d_\pm)\,\frac{\pi^2}{90}\,T^4$ of an ideal gas of photons, with the non-transverse degrees of freedom compensated by the ghost contribution, and electrons/positrons. Moreover, the leading-order perturbative result is reproduced since it originates, as noted above, solely from the Φ contribution. From the individual terms in (33), the boson and fermion contributions of order $\mathcal{O}(e^2)$ arise in analogy to (23) in the scalar theory,

$$\Omega^\star_{\text{lo}} = \frac{T^2}{24}\left([2-1]M_\gamma^2 + [2-1]M_e^2\right) = \Omega^{pert}_{\text{lo}}\,. \tag{34}$$

As in the calculation [15] of the HTL entropy, the leading-order contribution is therefore determined entirely by the behavior of the self-energies near the light cone, where the HTL approximation is, in fact, valid also asymptotically [19], which justifies *a posteriori* the usage of this approximation in the present approach[5]. However, in contrast to the entropy which is manifestly ultraviolet finite, the agreement of $\Omega^\star$ with the perturbative result is directly related to the cancellation of the thermal divergences. This aspect provides another, formal, argument for the approximation (32) of the Φ contribution: any other linear combination of traced boson and

---

[5]This also supports the phenomenological quasiparticle models applied (for QCD) in [5], which consider the thermodynamically relevant excitations as a gas of quasiparticles with effective masses equivalent to the asymptotic masses.

fermion self-energy contributions would result in either an uncompensated thermal divergence or an incorrect perturbative limit.

The next-to-leading order term of the perturbative expansion, as already discussed, cannot be expected to be reproduced in the present framework. Nevertheless, it is a noteworthy remark that

$$\Omega^{\star}_{\text{nlo}} = -[1 - 3/4] \frac{T}{12\pi} (2M_{\gamma}^2)^{3/2} , \tag{35}$$

which originates from the static longitudinal parts of the $\ln(-D^{-1})$ and the $D\Pi$ contribution in (33), as indicated by the bracket, underestimates the perturbative result $\Omega^{pert}_{\text{nlo}} = -T/(12\pi)(e^2T^2/3)^{3/2}$ again by a factor of $1/4$, for the same combinatorial reason as in the scalar theory, cf. (23). As for the scalar model, the $\mathcal{O}(e^3)$ term could be reproduced correctly by evaluating the two-loop Φ diagram with the dressed HTL propagators. Not surprisingly, this kind of approximation results also in the present case in an uncompensated thermal divergence $\propto (eT)^4$ and is, thus, relevant only as an expansion up to next-to-leading order.

The full expression (33) resums terms of all orders in the coupling constant and has to be evaluated numerically. The pressure $p^{\star} = -\Omega^{\star}$ turns out to be a monotonically decreasing function of the coupling [18], with the anticipated behavior of a physical approximation: it yields a smooth extrapolation of the leading-order perturbative result to the large-coupling regime, where it is enclosed between the leading and the next-to-leading order approximations which, as asymptotic expansions, presumably represent lower and upper boundaries for the exact result. For large values of e, the behavior of the pressure $p^{\star}$ is determined by terms $\propto M_{\gamma,e}^4 \propto (eT)^4$ which stem, as shown in the scalar theory, from the interplay between the quantum and thermal fluctuations[6]. The fermion contribution $\propto M_e^4$ is negative and overcompensates the positive photon contribution, and the total pressure $p^{\star}$ becomes negative when extrapolated beyond $e \sim 8.5$. While in this regime the resummed approximation is certainly not applicable, it will be argued in the following section that the boson contribution indicates the breakdown of the approximation already for a smaller coupling strength, at $\alpha = e^2/(4\pi) \sim 1.5$.

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[6]In the present case, parts of these terms are implicit in the individual quasiparticle and Landau-damping contributions, depending on the chosen subtraction terms, see Appendix.

4 QCD

In this section, I consider QCD, the original problem of interest. Within the framework of the HTL approximation, the quark propagator is analogous to the electron propagator in the Abelian plasma. Accordingly, the contribution of the quarks to the thermodynamic potential is analogous to the electron contribution in QED, so the following considerations focus on the gauge sector of QCD.

For a pure gauge plasma, the thermodynamic potential is expressed as a functional of the gluon propagator $D = (D_0^{-1} - \Pi)^{-1}$ and, in covariant gauges, the propagator $G = (G_0^{-1} - \Xi)^{-1}$ of the anticommuting bosonic ghost fields, by

$$\Omega = \frac{1}{2} \operatorname{Tr}[\ln(-D^{-1}) + D\Pi] - \operatorname{Tr}[\ln(-G^{-1}) + G\Xi] - \Phi \,, \tag{36}$$

where the summation over the color indices is implicit in the trace. With the same remark about gauge invariance as in the previous section, the functional Φ and the related self-energies are given to leading-loop order by

$$
\Phi_{ll} = -\frac{1}{12} \vcenter{\hbox{}} - \frac{1}{8} \vcenter{\hbox{}} + \frac{1}{2} \vcenter{\hbox{}} \,,
$$

$$
\Pi_{ll} = -\frac{1}{2} \vcenter{\hbox{}} - \frac{1}{2} \vcenter{\hbox{}} + \vcenter{\hbox{}} \,,
$$

$$
\Xi_{ll} = - \vcenter{\hbox{}} \,. \tag{37}
$$

In contrast to the photon polarization function, the gluon self-energy need not be transverse and it depends, in general, on four scalar functions. However, approximating the selfconsistent solutions of the coupled Dyson equations by their HTL contributions, as in QED, the gluon self-energy coincides with the Abelian expression (28) up to the replacement of M_γ^2 by the asymptotic gluon mass

$$M_g^2 = \frac{N_c}{6} g^2 T^2 \,, \tag{38}$$

with $N_c = 3$ for QCD. The ghost self-energy vanishes in the HTL approximation. Therefore, after rescaling the asymptotic mass and taking into account the color degrees of freedom, the contribution

$$(\Omega + \Phi)^\star = \frac{1}{2} \operatorname{Tr}[\ln(-D^{\star -1}) + D^\star \Pi^\star] - \operatorname{Tr}[\ln(-G_0^{-1})] \tag{39}$$

to the thermodynamic potential is equivalent to the respective photon-ghost contribution in QED. The Φ functional, however, has a more complicated topology than its Abelian counterpart, and requires a more detailed discussion.

Evaluated with the selfconsistent solutions of the coupled Dyson equations, the contributions of the individual graphs for Φ_{ll} could be expressed as

$$\frac{1}{6}\,\mathrm{Tr}D_{ll}\Pi_{ll}^{3\mathrm{g}}\,,$$

$$\frac{1}{4}\,\mathrm{Tr}D_{ll}\Pi_{ll}^{4\mathrm{g}}\,,$$

$$\frac{1}{2}\,\mathrm{Tr}D_{ll}\Pi_{ll}^{\mathrm{gh}} = -\frac{1}{2}\,\mathrm{Tr}G_{ll}\Xi_{ll}\,, \tag{40}$$

where, in obvious notation, the terms Π_{ll}^{i} denote the individual contributions to Π_{ll} in (37). Note that at this level of approximation, unlike in QED, Φ_{ll} cannot be represented as a linear combination of the gluon and ghost self-energies traced over their external momenta.

As for QED, the Φ contribution cannot be approximated by naively evaluating the expressions (40) within the HTL approximation, since the terms of higher order in the external momentum, which are neglected in the HTL approximation of the self-energies, are important in the traces over the whole phase space. Again, to keep track of these terms, the leading-order contributions of the individual expressions in (40), which are obtained by replacing the dressed propagators by bare ones, are now considered explicitly. The first term can be represented as a double sum-integral over an expression with a numerator $N = K^2 + \frac{1}{2}P^2$. Closing the external legs of the HTL contribution to $\Pi^{3\mathrm{g}}(P)$ amounts to neglecting the term $\frac{1}{2}P^2$ in N, thus

$$\frac{1}{6}\,\mathrm{Tr}D_{ll}\Pi_{ll}^{3\mathrm{g}}\Big|_{\mathrm{lo}} = \frac{3}{2}\frac{1}{6}\,\mathrm{Tr}D^{\star}\Pi^{3\mathrm{g}\star}\Big|_{\mathrm{lo}}\,. \tag{41}$$

The leading-order contribution of the diagram containing the gluon tadpole is saturated by the HTL contribution,

$$\frac{1}{4}\,\mathrm{Tr}D_{ll}\Pi_{ll}^{4\mathrm{g}}\Big|_{\mathrm{lo}} = \frac{1}{4}\,\mathrm{Tr}D^{\star}\Pi^{4\mathrm{g}\star}\Big|_{\mathrm{lo}}\,. \tag{42}$$

Finally, it is noted that the contributions omitted in the HTL approximation of the last two terms in (40) are taken into account in their sum, hence

$$
\begin{aligned}
\frac{1}{4}\,\mathrm{Tr}D_{ll}\Pi_{ll}^{\mathrm{gh}}\Big|_{\mathrm{lo}} - \frac{1}{4}\,\mathrm{Tr}G_{ll}\Xi_{ll}\Big|_{\mathrm{lo}} &= \frac{1}{4}\,\mathrm{Tr}D^{\star}\Pi^{\mathrm{gh}\star}\Big|_{\mathrm{lo}} - \frac{1}{4}\,\mathrm{Tr}G^{\star}\Xi^{\star}\Big|_{\mathrm{lo}} \\
&= \frac{1}{4}\,\mathrm{Tr}D^{\star}\Pi^{\mathrm{gh}\star}\Big|_{\mathrm{lo}}\,, \tag{43}
\end{aligned}
$$

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since the ghost self-energy vanishes in the HTL approximation. Combining the right hand sides of the expressions (41)-(43), the Φ contribution is approximated by

$$\Phi^\star = \frac{1}{4}\text{Tr}[D^\star(\Pi^{3g\star} + \Pi^{4g\star} + \Pi^{gh\star})] = \frac{1}{4}\text{Tr}D^\star\Pi^\star\,, \qquad (44)$$

which is gauge independent. As for the corresponding expression (32) in QED, the compact form of (44) is not a surprise. Rather, it reflects the fact that the combinatorial factors related to the number of propagator lines in the skeleton diagrams also account for how many of them can be soft in the HTL approximation.

The resummed approximation of the $SU(N_c)$ thermodynamic potential reads

$$\Omega^\star = \frac{1}{2}\text{Tr}\left[\ln(-D^{\star\,-1}) + \frac{1}{2}D^\star\Pi^\star\right] - \text{Tr}[\ln(-G_0^{-1})] \qquad (45)$$

which, as already stated for the contribution (39), resembles the photon-ghost part in QED in eqn. (33). Hence, the analogy of the photon and the gluon propagators in the HTL approximation becomes also apparent in the thermodynamical properties, as already anticipated for the fermion contribution. Therefore, the properties emphasized for the approximation (33) in QED hold also for (45) in QCD, namely, the parallels to the scalar case and the structure of the thermal divergences (which arise solely from the boson contribution). Accordingly, the expansion of (45) in $M_g^2/T^2 = \frac{1}{6}N_c g^2$ reproduces the leading-order term of the perturbative result

$$\Omega^{pert} = -2(N_c^2 - 1)\frac{\pi^2 T^4}{90}\left[1 - \frac{15}{8\pi^2}\frac{N_c\,g^2}{6} + \frac{15}{4\pi^3}\left(2\frac{N_c\,g^2}{6}\right)^{3/2} + \ldots\right], \qquad (46)$$

but underestimates the next-to-leading correction by a factor of $1/4$, as in the scalar theory[7]. The resummed pressure $p^\star = -\Omega^\star$, evaluated numerically from eqn. (45) and depicted in Fig. 3 as a function of the mass scale of the gluon self-energy, displays a striking resemblance to the scalar case shown in Fig. 2. At small coupling it matches the leading-order perturbative result, but it decreases less fast with increasing M_g^2. For larger coupling, the pressure is dominated by a contribution $\propto M_g^4$, which is positive as in the scalar theory and leads, eventually, to an increasing behavior of the approximation, with a minimum at

$$\bar{M}_g^2 \sim 3.1\,T^2\,, \quad p^\star_{\min} = p^\star(\bar{M}_g^2) \sim 0.82\,p_0\,. \qquad (47)$$

---

[7]Replacing $M_g^2 \to M_g^2/[1 + \frac{3}{\pi}(N_c/3)^{1/2}g]$, as suggested in [15], reproduces the next-to-leading order term in Ω^{pert} and might yield an improved resummed approximation for Ω at larger coupling strength, similar as in the scalar case.
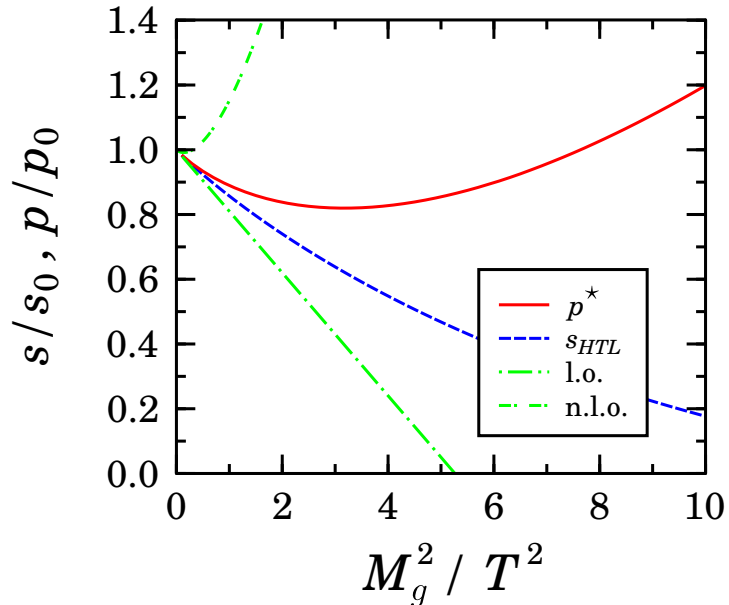
Figure 3: The pressure $p^\star$, from eqn. (45), and the entropy s_{HTL} [15] of the pure gauge plasma in the HTL approximation (scaled by the free values) as functions of the asymptotic mass. Depicted as well are the leading and next-to-leading order perturbative results as functions of $\frac{1}{6} N_c\, g^2 = M_g^2/T^2$.

These values compare to the numbers given in (16) for the scalar theory, which indicated in this simpler case, where the self-energy is just a mass term, that the approximation breaks down. Given the similarities of the expressions (22) and (45), the approximation for the thermodynamic potential in the $SU(N_c)$ theory, with a much more complex structure of the propagators, cannot be expected to be appropriate for couplings corresponding to asymptotic gluon masses larger than $\bar{M}_g$. The entropy s_{HTL} calculated in [15] from the leading loop entropy functional with the HTL propagators and depicted also in Fig. 3, on the other hand, shows no indication of a breakdown of the approximation at $\bar{M}_g$. Drawing the parallel to the scalar theory, however, the extrapolation of s_{HTL} cannot be physically meaningful beyond the point where the approximation of the pressure breaks down.

Taking an optimist's point of view, one may hope that the HTL-resummed pressure represents a reasonable approximation for $0.8\, p_0 \lesssim p^\star \leq p_0$, which, according to QCD lattice simulations [3, 4], corresponds to temperatures larger than $\bar{T} \sim 2.5\, T_c$.

This conjecture is supported by the observation that the HTL entropy, supplemented ('by hand') with the two-loop running coupling, is systematically off the $SU(3)$ lattice results [3] below $2\,T_c$, but starts to match the data for larger temperatures, just at $\bar{T}$. Therefore, the HTL approximations are presumably predictive even for moderate large coupling, where the conventional perturbative results are no longer meaningful. This allows to systematically address physically relevant questions, e.g., related to finite chemical potential[8], which can not yet be answered by QCD lattice calculations.

For even larger coupling, in the vicinity of the confinement transition, where finite-temperature lattice simulations predict a rapid change of the thermodynamic potential, more sophisticated approximations are needed. These may also clarify the question whether the qualitative change in the behavior of the thermodynamic potential, at about $2.5\,T_c$, is related to the breakdown of the leading-loop approximation at $\bar{T}$, which in the present approach remains a speculation.

5   Conclusions

In this paper, nonperturbative resummations of the thermodynamic potential have been considered, starting from the Φ-derivable approximation scheme at leading-loop order. By approximating the dressed propagators by their HTL contributions, at the expense of strict thermodynamical selfconsistency, physical and formally well-defined results have been obtained for the hot scalar theory, as well as for QED and QCD: they are ultraviolet finite, agree with the leading-order perturbative results and are, for gauge theories, explicitly gauge invariant, as are the approximations for the HTL-resummed entropy [15]. I stress, however, that the finiteness the approximate pressure is, due to cancellations of ultraviolet divergences of individual terms, a more subtle issue which does, on the other hand, provide a formal argument for the present approach.

Compared to the perturbative predictions, the HTL-resummed approximations display an improved behavior when extrapolated to larger coupling strength. For

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[8]By Maxwell's relation, as pointed out in [20] for the phenomenological quasiparticle models [5], the thermodynamical properties at finite temperature and at finite chemical potential are closely related, which allows us to infer, from the available finite-temperature lattice data, the equation of state of the QGP at nonzero baryon number.

all cases considered, at very large coupling strength, the resummed pressure is dominated by terms $\propto M^4$, where the mass scale M of the self-energies depends on the temperature and the coupling. These terms stem from the vacuum parts of the resummed contributions to the thermodynamic potential and lead, for bosons, to a pressure which increases as the coupling does. It has been shown for the scalar theory, and argued for gauge theories, that this feature is an indication of the breakdown of the leading-loop resummed approximation.

In QCD, this happens when the pressure is about 80% of the free limit, at a temperature $\bar{T} \sim 2.5\,T_c$. This explains the fact that the HTL-resummed entropy [15] starts to deviate systematically from the lattice results [3] below $2.5\,T_c$. On the other hand, the HTL approximation matches the lattice data for $T \gtrsim \bar{T}$. Therefore, the resummed approximations can be expected to be physical for rather large coupling, in a nonperturbative regime where the lattice simulations predict a saturation-like behavior of the thermodynamic potential. For even larger coupling, however, more sophisticated approximations are required for a detailed theoretical understanding of the thermodynamics of the QGP in the close vicinity of the confinement transition.

A   Appendix: The sum-integrals

In this appendix, the relevant boson sum-integrals are given explicitly in the $\overline{\text{MS}}$ regularization scheme in $d = 3 - 2\epsilon$ spatial dimensions.

Considering first the scalar theory, the vacuum and the thermal contributions of the trace $I(M^2, T)$ of the free propagator with mass M, defined in (7), are

$$
\begin{aligned}
I^0(M^2) &= \frac{M^2}{16\pi^2}\left[\frac{1}{\epsilon} + \ln\frac{\bar{\mu}^2}{M^2} + 1\right], \\
I^T(M^2, T) &= -\int\frac{d^3k}{(2\pi)^3}\frac{n_b(\omega_M/T)}{\omega_M},
\end{aligned}
\tag{48}
$$

where $\omega_M = (k^2 + M^2)^{1/2}$. The contributions of the function $J = \frac{1}{2}\,\mathrm{tr}\ln(M^2 - K^2)$ introduced in (11), which is related to I by differentiation with respect to M^2, read

$$
\begin{aligned}
J^0(M^2) &= \frac{M^4}{64\pi^2}\left[-\frac{1}{\epsilon} - \ln\frac{\bar{\mu}^2}{M^2} - \frac{3}{2}\right], \\
J^T(M^2, T) &= \int\frac{d^3k}{(2\pi)^3}\,T\ln(1 - \exp[-\omega_M/T]).
\end{aligned}
\tag{49}
$$

For QED and QCD, apart from the so-called quasiparticle contributions originating from the poles $\omega_i(k)$ of the propagators, there are Landau-damping contributions from the imaginary parts of the self-energies, i.e., from below the light cone in the HTL approximation. The individual contributions are regularized by appropriate subtraction terms. The expressions given below apply for both gauge theories (in QCD, the color trace yields just a factor), with M_b being the respective asymptotic boson mass. The expression

$$\frac{1}{2} \sum_{i=T,L,gh} d_i \sum\!\!\!\!\!\!\!\int \ln(-\Delta_i^{-1\,\star})$$

$$= \sum_{i=T,L,gh} d_i \int_{k^3} \left[\frac{\omega_i}{2} + T \ln(1 - \exp[-\omega_i/T]) + \int_0^k \frac{d\omega}{2\pi} (1 + 2n_b)\phi_i - A_i^{sub} \right]$$

$$+ \frac{M_b^4}{32\pi^2} \left[-\frac{1}{\epsilon} - \ln \frac{\bar{\mu}^2}{M_b^2} - \frac{13}{9} + \frac{\pi^2}{3} - \frac{2}{9} \ln 2 + \frac{4}{3} \ln^2 2 \right] \tag{50}$$

is analogous to the function J in the scalar theory, cf. (49). In particular, the relevant number of degrees of freedom is apparent in both the thermal (quasiparticle) and in the divergent vacuum contribution. In (50), the contribution of the ghost fields is, formally, included as another degree of freedom with $d_{gh} = -1$ and $\omega_{gh}(k) = k$. The angles ϕ_i are defined as $\phi_{gh} \equiv 0$ and $\phi_{T,L} = \mathrm{Disc}\ln(-\Delta_{T,L}^{-1\,\star})$, and the infrared finite subtraction terms were chosen as $A_{gh} = 0$ and

$$\sum_{i=T,L} d_i A_i^{sub} = -d_T \left[\frac{k}{2} + \frac{M_b^2}{4k} + \frac{M_b^4}{8k(k^2 + M_b^2)} \left(\frac{3}{2} - \ln \frac{4(k^2 + M_b^2)}{M_b^2} \right) \right]$$

$$- \int_0^k \frac{d\omega}{2\pi} \mathrm{Im}\tilde{\Pi} \left[\frac{-d_T}{K^2 - M_b^2} \left(1 + \frac{\mathrm{Re}\tilde{\Pi}}{K^2 - M_b^2} \right) + \frac{2}{K^2} \left(1 - 2 \frac{\mathrm{Re}\tilde{\Pi}}{K^2} \frac{k^2}{k^2 + M_b^2} \right) \right].$$

The contribution resembling $\frac{1}{2} M^2 I$ in the scalar theory is

$$\frac{1}{2} \sum_{i=T,L} d_i \sum\!\!\!\!\!\!\!\int \Delta_i^\star \Pi_i^\star$$

$$= \sum_{i=T,L} d_i \int_{k^3} \left[-\frac{1}{2} \frac{(1 + 2n_b)\Pi_i^\star}{2\omega - \partial_\omega \Pi_i^\star} \bigg|_{\omega_i} + \int_0^k \frac{d\omega}{2\pi} (1 + 2n_b)\psi_i - B_i^{sub} \right]$$

$$+ \frac{M_b^4}{32\pi^2} \left[\frac{2}{\epsilon} + 2\ln \frac{\bar{\mu}^2}{M_b^2} + \frac{14}{9} - \frac{2\pi}{3} + \frac{16}{9} \ln 2 - \frac{8}{3} \ln^2 2 \right], \tag{51}$$

with $\psi_{T,L} = \mathrm{Disc}(\Delta_{T,L}^\star \Pi_{T,L}^\star)$ and

$$\sum_{i=T,L} d_i B_i^{sub} = -d_T \left(\frac{M_b^2}{4k} + \frac{M_b^4}{4k(k^2 + M_b^2)} \left(2 - \ln \frac{4(k^2 + M_b^2)}{M_b^2} \right) \right)$$

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$$-\int_0^k \frac{d\omega}{2\pi} \operatorname{Im}\tilde{\Pi} \left[\frac{-d_T K^2}{(K^2 - M_b^2)^2} \left(1 + 2\frac{\operatorname{Re}\tilde{\Pi}}{K^2 - M_b^2} \right) + \frac{2}{K^2} \left(1 - 4\frac{\operatorname{Re}\tilde{\Pi}}{K^2} \frac{k^2}{k^2 + M_b^2} \right) \right].$$

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